Bloch-Nordsieck Approximation in Linearized Quantum Gravity*

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Abstract

The aim of this article is to review Fradkin's contribution in the realm of eikonal physics. In particular, the so-called Fradkin representation is employed to investigate a certain subclass of Feynman diagrams resulting in an expression for the scattering amplitude at Planckian energies. The 't Hooft poles are reproduced.

1 Introduction

It is not difficult for me to give a tribute to E.S. Fradkin's scientific achievements because I already became acquainted with his work during the second half of the sixties. I was then a graduate student at Brown University, where Herb Fried lectured in his quantum field theory course on the Bloch-Nordsieck approximation and the Schwinger-Fradkin representation. The representation invented by Fradkin first appeared in a 1966 Nuclear Physics article [1]. It became the backbone of many of the eikonal articles in the years to come. The state of the art has been extensively reviewed in Ref. [2]. Similar methods have been adopted recently in [3] in order to study Planckian-energy gravitational scattering. I will pick up the story in 1970, when I published a short article in Phys. Rev. [4] which is concerned with the summation of a certain subclass of Feynman diagrams within the Bloch-Nordsieck (or eikonal) approximation. While this approximation was used in scalar and spinor QED at that time [5], it was not until 1992 that the same methods were applied again to linearized quantum gravity [6]. Now path integral techniques were en vogue and had long since overtaken functional differentiation methods as advocated by Schwinger-Symanzik or Fradkin himself.

However, in the sequel I will demonstrate within linearized quantum gravity that there is no need to go beyond the functional differentiation techniques that were already available in the sixties.

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2 Linearized Quantum Gravity with Spin-$\frac{1}{2}$ Matter Interaction

At the beginning of this chapter we write down the linearized Einstein-Hilbert action coupled to a spin-$\frac{1}{2}$ matter field. After the metric $g_{\mu\nu}$ has been expanded about a Minkowski flat background, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the gauge-fixed linearized action is given by

$$S = \frac{1}{16\pi G} \int d^4x \, \frac{1}{2} \left[\frac{1}{2} h_{\mu\nu} \partial^2 h^{\mu\nu} - \frac{1}{4} h \partial^2 h + 16\pi G \, T^{\mu\nu} h_{\mu\nu} \right]. \tag{1}$$

In the gauge-fixing term $-\frac{1}{2}C_\mu^2$ we have chosen the DeDonder gauge $C_\mu = \partial_\nu h_\mu^\nu - \frac{1}{2}\partial_\mu h$, where $h = h_\lambda^\lambda$ and all indices are to be raised and lowered with $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. As usual we set $\hbar = 1 = c$. The linearized matter action in Eq. (1) makes use of

$$T_{\mu\nu} = \frac{1}{4} \left\{ \bar{\psi} \gamma_\mu \left(\frac{1}{\mathrm{i}} \partial_\nu \psi \right) - \left(\frac{1}{\mathrm{i}} \partial_\nu \bar{\psi} \right) \gamma_\mu \psi + \left(\mu \leftrightarrow \nu \right) \right\}. \tag{2}$$

Note that the gravitational part in Eq. (1) leads to the graviton propagator

$$D_{\mu\nu\lambda\sigma}(k) = \frac{16\pi G}{k^2 - \mathrm{i}\epsilon} \left(\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma} \right). \tag{3}$$

Writing the Lagrangian of the matter interaction as

$$\mathcal{L}(\psi, \bar{\psi}, h_{\mu\nu}) = -\bar{\psi} \left(\gamma \frac{1}{\mathrm{i}} \partial + m \right) \psi + \frac{1}{2} h^{\mu\nu} T_{\mu\nu},$$

we obtain the Green's function equation from the action principle,

$$\left[m + \gamma \frac{1}{\mathrm{i}} \partial^x - \frac{1}{4} \gamma_\mu \left(\frac{1}{\mathrm{i}} \partial_\nu^x h^{\mu\nu}(x) + h^{\mu\nu}(x) \frac{1}{\mathrm{i}} \partial_\nu^x \right) \right] G(x, y | h_{\mu\nu}) = \delta(x - y). \tag{4}$$

There are several methods to solve Eq. (4) approximately. Since we are interested in elastic forward scattering processes with the exchange of an arbitrary number of soft gravitons, we proceed to evaluate Eq. (4) using the Bloch-Nordsieck (BN) or eikonal approximation. For this reason we begin by performing a semi-Fourier transform,

$$G(x, y | h_{\mu\nu}) = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{e}^{\mathrm{i}px} \, G(p, y | h). \tag{5}$$

In this way we obtain the following expression:

$$\left[m + \gamma \frac{1}{\mathrm{i}} \partial^x \right] G(x, y | h) = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{e}^{\mathrm{i}px} \left(\mathrm{e}^{-\mathrm{i}py} + \frac{1}{2} \gamma_\mu h^{\mu\nu}(x) p_\nu \, G(p, y | h) \right). \tag{6}$$

In this equation we have dropped the term $\frac{1}{4}\gamma_\mu \frac{1}{\mathrm{i}}\partial^x_\nu h^{\mu\nu}(x) = \frac{1}{8}\gamma_\mu \frac{1}{\mathrm{i}}\partial^\mu h(x)$ which leads to a kind of self-coupling – the same index μ.

When we now perform the BN-approximation in Minkowski flat space by setting

$$\begin{aligned} \gamma_\mu &\rightarrow \langle\gamma_\mu\rangle = v_\mu = \frac{p_\mu}{m}, \qquad v^2 = -1, \\ p_\nu &\simeq p'_\nu \simeq \frac{p_\nu + p'_\nu}{2}, \end{aligned} \tag{7}$$

we observe that it is exactly the second term on the right-hand side of Eq. (6) that gives us the desired $p_\mu h^{\mu\nu} p_\nu$-interaction term. Within this approximate treatment we now have to solve the Green's function equation,

$$\left[m + v\frac{1}{\mathrm{i}}\partial - \frac{1}{2}v_\mu p_\nu h^{\mu\nu}(x)\right] G_{\mathrm{BN}}(x, y|h) = \delta(x-y). \tag{8}$$

At this stage we first introduce Schwinger's proper-time representation [7] (we follow the pattern of calculations as outlined in Ref. [8] for a scalar field theory):

$$\begin{aligned} G_{\mathrm{BN}}[h] &= \mathrm{i}\int_0^\infty ds\, \mathrm{e}^{-\mathrm{i}s\left[m - \mathrm{i}\epsilon + v\frac{1}{\mathrm{i}}\partial + B\right]}, \qquad B := -\frac{1}{2}v_\mu p_\nu h^{\mu\nu} \\ &= \mathrm{i}\int_0^\infty ds\, \mathrm{e}^{-\mathrm{i}s(m-\mathrm{i}\epsilon)} \mathrm{e}^{\mathrm{i}s[v\mathrm{i}\partial - B]} = \mathrm{i}\int_0^\infty ds\, \mathrm{e}^{-\mathrm{i}s(m-\mathrm{i}\epsilon)} U(s), \end{aligned} \tag{9}$$

where $U(s) = \mathrm{e}^{\mathrm{i}s[v\mathrm{i}\partial - B]}$. In the second step we follow Fradkin by coupling the operator $(v\cdot\mathrm{i}\partial)$ to the "source" $\nu(s)$. This is done by introducing

$$U(s,\nu) = T_{s'}\left(\mathrm{e}^{\mathrm{i}\int_0^s ds'\,(v\cdot\mathrm{i}\partial - B + \nu(s')(v\cdot\mathrm{i}\partial))}\right) \tag{10}$$

with the property $U(s,\nu)|_{\nu\rightarrow 0} = U(s)$. If we now recall the simple formula

$$F\left\{\frac{\delta}{\delta\nu}\right\} \mathrm{e}^{\mathrm{i}\int \nu f} = F\{\mathrm{i}f\}\, \mathrm{e}^{\mathrm{i}\int \nu f}$$

or

$$\mathrm{e}^{\int_0^s ds'\, \frac{\delta}{\delta\nu(s')}}\, \mathrm{e}^{\mathrm{i}\int_0^s ds'\, \nu(s')(v\cdot\mathrm{i}\partial)} = \mathrm{e}^{\mathrm{i}\int_0^s ds'\,(v\cdot\mathrm{i}\partial)}\, \mathrm{e}^{\mathrm{i}\int_0^s ds'\nu(s')(v\cdot\mathrm{i}\partial)}, \tag{11}$$

we obtain

$$U(s,\nu) = \mathbb{1}\,\mathrm{e}^{\int\limits_0^{s-\epsilon} ds'\,\frac{\delta}{\delta\nu(s')}}\,T_{s'}\left(\mathrm{e}^{\int\limits_0^{s} ds'\left[-\nu(s')(v\cdot\partial)-\mathrm{i}B\right]}\right). \tag{12}$$

The expressions (10) and (12) imply

$$\begin{aligned} \frac{\partial U(s,\nu)}{\partial s} &= \mathrm{i}\Big[v\cdot\mathrm{i}\partial - B + \nu(s)(v\cdot\mathrm{i}\partial)\Big]U(s,\nu), \qquad U(0,0)=1 & (13)\\ \text{and}\qquad \frac{1}{\mathrm{i}}\frac{\delta U(s,\nu)}{\delta\nu(s)} &= (v\cdot\mathrm{i}\partial)U(s,\nu). & (14)\end{aligned}$$

At this point it is convenient to define two new functions $W(s)$ and $f(s)$ according to

$$\begin{aligned} W(s) &\equiv T_{s'}\left(\mathrm{e}^{\int\limits_0^{s} ds'\,[-\nu(s')(v\cdot\partial)-\mathrm{i}B]}\right), \\ W(s) &= \mathrm{e}^{-\int\limits_0^{s} ds'\,\nu(s')(v\cdot\partial)}\,f(s), & (15)\\ \text{with}\qquad f(s) &= \mathrm{e}^{\int\limits_0^{s} ds'\,\nu(s')(v\cdot\partial)}\,W(s). & (16)\end{aligned}$$

It is then easy to set up a differential equation for $\frac{\partial f}{\partial s}$ with $f(0)=1$ whose solution is given by

$$\langle x|f(s)|y\rangle = \mathrm{e}^{-\mathrm{i}\int\limits_0^{s} ds'\,B\left(x+\int\limits_0^{s'} ds''\,\nu(s'')\,v\right)}\,\langle x|y\rangle. \tag{17}$$

When this expression is substituted into Eq. (15) we obtain

$$\langle x|W(s)|y\rangle = \mathrm{e}^{-\mathrm{i}\int\limits_0^{s} ds'\,B\left(y+\int\limits_0^{s'} ds''\,\nu(s'')v\right)}\,\delta\left(x-y-\int\limits_0^{s} ds'\,\nu(s')v\right). \tag{18}$$

This result enables us to write our solution in the form

$$\begin{aligned} G_{\mathrm{BN}}(x,y|h) &= \mathrm{i}\int\limits_0^{\infty} ds\,\mathrm{e}^{-\mathrm{i}sm}\mathrm{e}^{\int\limits_0^{s} ds'\,\frac{\delta}{\delta\nu(s')}}\,\mathrm{e}^{\mathrm{i}\frac{p_\mu p_\nu}{2m}\int\limits_0^{s} ds'\,h_{\mu\nu}\left(y+\int\limits_0^{s'} ds''\,\nu(s'')v\right)} \\ &\qquad\times\delta\left(x-y-\int\limits_0^{s} ds'\,\nu(s')v\right)\Bigg|_{\nu=0}. & (19)\end{aligned}$$

A simple functional Fourier transform yields

$$\mathrm{e}^{\int_0^s ds' \, \frac{\delta}{\delta\nu(s')}} F\{\nu\}\Big|_{\nu=0} = \mathcal{N}\int \mathcal{D}\phi \, F\{\phi\} \, \mathrm{e}^{\mathrm{i}\int_0^s ds' \, \phi(s')} \, ,$$

so that an equivalent path integral representation of Eq. (19) is given by

$$\begin{aligned} G_{\mathrm{BN}}(x,y|h) &= \mathrm{i}\int_0^\infty ds \, \mathrm{e}^{-\mathrm{i}sm}\mathcal{N}(s)\int \mathcal{D}\phi \, \mathrm{e}^{\mathrm{i}\int_0^s ds' \, \phi(s')} \, \mathrm{e}^{\mathrm{i}\frac{p_\mu p_\nu}{2m}\int_0^s ds' \, h_{\mu\nu}\left(y+\int_0^{s'} ds'' \, \phi(s'')v\right)} \\ &\quad \times\delta\left(x-y-\int_0^s ds' \, \phi(s')v\right) \, , \end{aligned} \tag{20}$$

with $\mathcal{N}(s)^{-1} = \int \mathcal{D}\phi \, \exp\left(\mathrm{i}\int_0^s ds' \, \phi(s')\right)$. For similar formulas in QED, see Ref. [9].

If we now, in the eikonal spirit, restrict the sum over paths to straight-line paths between y and x, we obtain from Eq. (19)

$$G_{\mathrm{BN}}(x,y|h) = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{e}^{\mathrm{i}p(x-y)}\mathrm{i}\int_0^\infty ds \, \mathrm{e}^{-\mathrm{i}s(m+vp)}\mathrm{e}^{\mathrm{i}\frac{p^\mu p^\nu}{2m}\int_0^s ds' \, h_{\mu\nu}\left[x\frac{s'}{s}+\left(1-\frac{s'}{s}\right)y\right]} . \tag{21}$$

Introducing the variable $u = \frac{s'}{s}$, the s integration in Eq. (21) is easily performed:

$$\int \frac{d^4p}{(2\pi)^4} \, \mathrm{e}^{\mathrm{i}p(x-y)}\left\{\left[m-\frac{p^\mu p^\nu}{2m}\int_0^1 du \, h_{\mu\nu}(xu+(1-u)y)\right]+vp\right\}^{-1} . \tag{22}$$

This representation, compact as it is, is, however, difficult to use in momentum space. Fortunately Eq. (21) can also be rewritten in the form

$$G_{\mathrm{BN}}(x,y|h) = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{e}^{\mathrm{i}p(x-y)} \, \mathrm{i}\int_0^\infty ds \, \mathrm{e}^{-\mathrm{i}s(m+vp)}\mathrm{e}^{\frac{\mathrm{i}}{2}\frac{p^\mu p^\nu}{m}\int_0^s ds' \, h_{\mu\nu}\left(x-\frac{p}{m}s'\right)} . \tag{23}$$

It is this representation of the eikonalized Green's function which we will employ in the sequel when computing scattering amplitudes.

3 Green's Function, Newton Potential

In this section we are interested in the four-point Green's function where we treat the two-point Green's function for particle #1 in the BN-approximation while the Green's function

for particle #2 is the exact one, i.e., satisfies Eq. (4). At this point we need the on-shell truncated, connected Green's function for particle #1:

$$G^{\rm c}_{\rm BN}(p,p'|h) = \lim_{m+vp=0,\, m+vp'=0} (m+vp)\, G_{\rm BN}(p,p'|h)\, (m+vp'), \tag{24}$$

where the momentum-space version of Eq. (23) is given by

$$G_{\rm BN}(p,p'|h) = \int d^4x\, {\rm e}^{-{\rm i}(p-p')x}\, {\rm i}\int\limits_0^\infty ds\, {\rm e}^{-{\rm i}s(m+vp')} {\rm e}^{\frac{\rm i}{2} v^\mu p'^\nu \int\limits_0^s ds'\, h_{\mu\nu}(x-vs')}. \tag{25}$$

Two integrations by parts on the variable s then lead to

$$G^{\rm c}_{\rm BN}(p,p'|h) = -{\rm i}\int d^4x\, {\rm e}^{-{\rm i}(p-p')x} \frac{\partial}{\partial s}\left({\rm e}^{{\rm i}\frac{p^\mu p^\nu}{2m}\int\limits_{-\infty}^{s} ds'\, h_{\mu\nu}\left(x+\frac{p}{m}s'\right)}\right)_{s=0}. \tag{26}$$

Making a change of variables,

$$x = z + \frac{p}{m}\beta,$$

where z is space-like ($-z^2 < 0$) and p time-like ($-p^2 > 0$), we find

$$G^{\rm c}_{\rm BN}(p,p'|h) = -{\rm i}\frac{p^0}{m}\int d^3\mathbf{z}\, {\rm e}^{{\rm i}(p'-p)z}\, {\rm e}^{{\rm i}\frac{p^\mu p^\nu}{2m}\int\limits_{-\infty}^{\infty} d\tau\, h_{\mu\nu}\left(z+\frac{p}{m}\tau\right)}. \tag{27}$$

Since we are dealing with spinors we must not forget to sandwich Eq. (27) between $\bar{u}_\alpha(p)\ldots u_\beta(p') \to \delta_{\alpha\beta}$, so that Eq. (27) turns into

$$G^{\rm c}_{\rm BN}(p,p'|h) \;=\; -{\rm i}\frac{p^0}{m}\,\delta_{\alpha\beta}\int d^3\mathbf{z}\, {\rm e}^{{\rm i}(p'-p)z}\, {\rm e}^{{\rm i}\frac{p^\mu p^\nu}{2m}\int\limits_{-\infty}^{\infty} d\tau\, h_{\mu\nu}\left(z+\frac{p}{m}\tau\right)}, \tag{28}$$

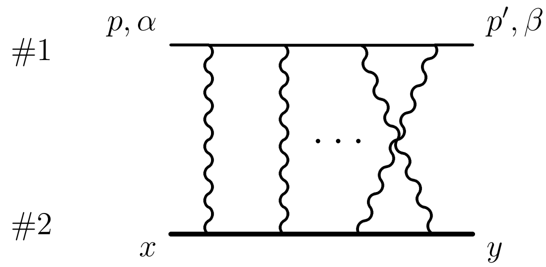
For further studies it is convenient to rewrite Eq. (28) in the form

$$\langle p,\alpha|G^{\rm c}_{\rm BN}[h]|p',\beta\rangle \;=\; -{\rm i}\frac{p^0}{m}\,\delta_{\alpha\beta}\int d^3\mathbf{z}\, {\rm e}^{{\rm i}(p'-p)z}\, {\rm e}^{\frac{\rm i}{2}\int d^4\xi\, T^{\mu\nu}(\xi) h_{\mu\nu}(\xi)}, \tag{29}$$

$$T^{\mu\nu}(\xi) \;:=\; \int\limits_{-\infty}^{\infty} d\tau\, \frac{1}{m}\, p^\mu p^\nu\, \delta^4\left(\xi - \left(z + \frac{p}{m}\tau\right)\right), \tag{30}$$

where $T^{\mu\nu}$ is the classical energy–momentum tensor for a point particle with momentum p.

Now let us return to our forward scattering process viewed in a system where particle #1 is highly relativistic compared to particle #2. Then the truncated scattering amplitude is given by



$$
\begin{aligned}
\langle x, p\alpha | y, p'\beta\rangle &\equiv G(x, p\alpha; y, p'\beta|h) \\
&= \mathrm{e}^{-\mathrm{i}\frac{\delta}{\delta h_1^{\alpha\beta}} D^{\alpha\beta\gamma\delta} \frac{\delta}{\delta h_2^{\gamma\delta}}} \, G^{\mathrm{c}}_{\mathrm{BN}1}(p, \alpha, p', \beta|h_1)\, G_2(x, y|h_2)\Big|_{h_{1\mu\nu}=0=h_{2\mu\nu}} \\
&= \mathrm{e}^{-\mathrm{i}\frac{\delta}{\delta h_1^{\alpha\beta}} D^{\alpha\beta\gamma\delta} \frac{\delta}{\delta h_1^{\gamma\delta}}} \int d^4x_1\, \mathrm{e}^{\mathrm{i}(p'-p)x_1}\, (-\mathrm{i}\delta_{\alpha\beta}) \\
&\quad \times \frac{\partial}{\partial \tau_1}\left(\mathrm{e}^{\mathrm{i}\frac{p^\mu p^\nu}{2m}\int_{-\infty}^{\tau_1} d\tau_1'\, h_{1\mu\nu}\left(x_1 + \frac{p}{m_1}\tau_1'\right)} \right)_{\tau_1=0} G_2(x, y|h_2)\Big|_{h_{1\mu\nu}=0=h_{2\mu\nu}} .
\end{aligned}
\tag{31}
$$

The final result of the functional differentiation in Eq. (31) is

$$
\begin{aligned}
\langle p, \alpha| \big(\psi(x)\bar{\psi}(y)\big)_+ |p', \beta\rangle^{h_{\mu\nu}} &= G_{\mathrm{BN}}(x, p\alpha; y, p'\beta|h) \\
&= -\mathrm{i}\frac{p^0}{m}\,\delta_{\alpha\beta}\int d^3\mathbf{z}\, \mathrm{e}^{\mathrm{i}(p'-p)z}\, G_2\Bigg(x, y\Bigg| \underbrace{\frac{p^\mu p^\nu}{2m_1}\int_{-\infty}^{\infty} d\tau\, D^{\mu\nu\alpha\beta}\left(z + \frac{p}{m_1}\tau - \xi\right)}_{=h_1^{\alpha\beta}\left(z + \frac{p}{m_1}\tau - \xi\right)}\Bigg),
\end{aligned}
\tag{32}
$$

$$
\text{with} \quad h_{1\mu\nu}(\xi) = \frac{1}{2}\int d^4\xi'\, D_{\mu\nu\alpha\beta}(\xi - \xi')\, T_1^{\alpha\beta}(\xi').
$$

Formula (32) expresses the four-point function in terms of the Green's function of the spinor particle #2 in presence of the linearized gravitational background field $h_{1\mu\nu}$ generated by the (eikonalized) incoming particle #1.

We can rewrite our result (32) as

$$
G_{\mathrm{BN}} = -\mathrm{i}\frac{p^0}{m}\,\delta_{\alpha\beta}\int d^3\mathbf{z}\, \mathrm{e}^{\mathrm{i}(p'-p)z}\, G_2\left(x+z, y+z\Bigg| \frac{p^\mu p^\nu}{2m_1}\int_{-\infty}^{\infty} d\tau\, D^{\mu\nu\alpha\beta}\left(\xi - \frac{p}{m_1}\tau\right)\right).
$$

Using

$$D_{\mu\nu\alpha\beta}\left(\xi - \frac{p}{m_1}\tau\right) = 16\pi G \int \frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{\mathrm{i}k\left(\xi-\frac{p}{m_1}\tau\right)}}{k^2-\mathrm{i}\epsilon}\,\big(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}\big),$$

we find for the functional argument of G_2:

$$\frac{8\pi G}{m_1}\big(2p_\alpha p_\beta - \underbrace{p^2}_{-m_1^2}\eta_{\alpha\beta}\big)\int\limits_{-\infty}^{\infty} d\tau\, D\left(\xi-\frac{p}{m_1}\tau\right),$$

or:

$$\begin{aligned} G_{\mathrm{BN}}(x,p\alpha;y,p'\beta|h) &= -\mathrm{i}\frac{p^0}{m}\,\delta_{\alpha\beta}\int d^3\mathbf{z}\,\mathrm{e}^{\mathrm{i}(p'-p)z} \\ &\quad\times G_2\left(x+z,y+z\middle|8\pi G\big(2p_\alpha p_\beta - p^2\eta_{\alpha\beta}\big)\int\limits_{-\infty}^{\infty} d\tau\, D\left(\xi-p\tau\right)\right). \end{aligned} \tag{33}$$

Now let us go to the rest frame of particle #1:

$$p=(p^0,\mathbf{p})\equiv(m_1,\mathbf{0}).$$

Then the functional argument of G_2 in Eq. (33) turns into ($\eta^{00}=-1$)

$$\begin{aligned} &8\pi G(2m_1^2-m_1^2)\underbrace{\int\limits_{-\infty}^{\infty} d\tau\, D(m_1\tau-\xi^0,\mathbf{0}-\boldsymbol{\xi})}_{\overset{m_1\tau=t}{=}\frac{1}{m_1}\underbrace{\int\limits_{-\infty}^{\infty} dt\, D(t-\xi^0,-\boldsymbol{\xi})}_{=\frac{1}{4\pi|\boldsymbol{\xi}|},\quad |\boldsymbol{\xi}|=r}}, \\ &= \frac{2Gm_1}{r}. \end{aligned} \tag{34}$$

Recall that the functional argument of G_2 in Eq. (33) is also the linearized external soft gravitational field $h_1^{\alpha\beta}$:

$$h_1^{\alpha\beta} = 8\pi G(2p^\alpha p^\beta + m_1^2\eta^{\alpha\beta})\int\limits_{-\infty}^{\infty} d\tau\, D(\xi-p\tau).$$

So we find for

$$h^{00} = \big(h_{00} =\big)\, \frac{2Gm}{r}. \tag{35}$$

Likewise, $h_{jj} = 8\pi G m_1^2 \frac{1}{m_1}\left(\frac{1}{4\pi r}\right)$ or

$$h_{jj} = \frac{2Gm_1}{r}, \tag{36}$$

i.e., if we choose to work in the rest frame of particle #1, this gives rise – in linearized gravity – to the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

$$\begin{aligned} (ds)^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu \\ &\overset{\text{lin.}}{=} (-1 + h_{00})\, dt^2 + (1 + h_{jj})\, d\mathbf{x}^2 + \dots, \end{aligned}$$

where we have set $c = 1$. So we obtain

$$(ds)^2 = \left(-1 + \frac{2Gm_1}{r}\right) dt^2 + \left(1 + \frac{2Gm_1}{r}\right) \left(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right) \tag{37}$$

instead of the square of the line element in the Schwarzschild metric:

$$(ds)^2 = -\left(1 - \frac{2Gm_1}{r}\right) dt^2 + \frac{1}{1 - \frac{2Gm_1}{r}}\, dr^2 + r^2 d\Omega^2,$$

where we have abbreviated $d\theta^2 + \sin^2\theta\, d\phi^2$ by $d\Omega^2$.

4 Scattering Amplitude, 't Hooft Poles

Finally we turn to the elastic scattering of two electrons which exchange gravitons in all possible ladder-type ways. As long as self-energy structure and vertex corrections are neglected, we need to evaluate the four-point Green's function ($q_1 + q_2 \to p_1 + p_2$):

$$\begin{aligned} \langle q_1 q_2 | p_1 p_2 \rangle &\equiv M(q_1, q_2; p_1, p_2) \\ &= \mathrm{e}^{-\mathrm{i} \frac{\delta}{\delta h_1^{\alpha\beta}} D^{\alpha\beta\gamma\delta} \frac{\delta}{\delta h_2^{\gamma\delta}}}\, G_{\mathrm{BN}}[h_{1\mu\nu}]\, G_{\mathrm{BN}}[h_{2\mu\nu}]\Big|_{h_{1\mu\nu}=0=h_{2\mu\nu}} \qquad (38) \\ &= -4 m_1 m_2 \int d^4x_1\, d^4x_2\, \mathrm{e}^{-\mathrm{i}(q_1 - p_1)x_1} \mathrm{e}^{-\mathrm{i}(q_2 - p_2)x_2} \\ &\quad \times \left(\frac{\partial}{\partial \tau_1} \frac{\partial}{\partial \tau_2} \mathrm{e}^{\mathrm{i} \frac{1}{2m_1} \frac{1}{2m_2} q_2^\alpha q_2^\beta q_1^\gamma q_1^\delta \int_{-\infty}^{\tau_1} d\tau_1' \int_{-\infty}^{\tau_2} d\tau_2'\, D_{\alpha\beta\gamma\delta}\left(x_2 - x_1 + \frac{q_2}{m_2}\tau_2' - \frac{q_1}{m_1}\tau_1'\right)} \right)_{\tau_1 = 0 = \tau_2}. \qquad (39) \end{aligned}$$

Since we are mainly interested in the pole structure of the scattering amplitude, we drop the non-spin-flip factors $\delta_{\alpha\beta}\delta_{\gamma\delta}$. Expression (39) can best be calculated in the CM system, where it yields ($q = q_1 - p_1$, $q' = q_2 - p_2$):

$$
\begin{aligned}
M(q,q') &= -4m_1m_2(2\pi)^4\,\delta(q+q')\int d^4\xi\,\mathrm{e}^{\mathrm{i}q\xi} \\
&\quad\times\left(\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}\mathrm{e}^{\frac{\mathrm{i}}{4m_1m_2}16\pi G\left(2(q_1q_2)^2-m^4\right)\int\limits_{-\infty}^{\tau_1}d\tau_1'\int\limits_{-\infty}^{\tau_2}d\tau_2'\,D\left(\xi+\frac{q_1}{m_1}\tau_1'-\frac{q_2}{m_2}\tau_2'\right)}\right)_{\tau_1=0=\tau_2}. \qquad (40)
\end{aligned}
$$

In the CM system, $\mathbf{q_2} = -\mathbf{q_1}$, we have $s = -(q_1+q_2)^2 = 2m^2 - 2q_1q_2$, so that

$$
2(q_1q_2)^2 - m^4 = \frac{1}{2}[(s-2m^2)^2 - 2m^4] =: \gamma(s). \qquad (41)
$$

We also introduce four-vectors,

$$
\begin{aligned}
u^\mu_{1,2} &= \frac{1}{\sqrt{2}}(1,0,0,\pm1), \qquad u_1^2 = 0 = u_2^2, \qquad u_{1\mu}u_2^\mu = -1, \\
\text{so that} \qquad & q_1^\mu q_{2\mu} = -\frac{s}{2}, \qquad (42)
\end{aligned}
$$

following from

$$
\begin{aligned}
q_1^\mu &= \left(\sqrt{\mathbf{q_1}^2+m_1^2},0,0,q_1\right) \overset{\mathbf{q}_1\to\infty}{\rightarrow} (q_1,0,0,q_1) = \frac{\sqrt{s}}{2}(1,0,0,1), \\
q_2^\mu &= \left(\sqrt{\mathbf{q_1}^2+m_2^2},0,0,-q_1\right) \overset{\mathbf{q}_1\to\infty}{\rightarrow} (q_1,0,0,-q_1) = \frac{\sqrt{s}}{2}(1,0,0,-1).
\end{aligned}
$$

A decomposition of ξ^μ in $\xi^\mu = \xi_\mathrm{T}^\mu + \xi_1u_1^\mu + \xi_2u_2^\mu$, where the components ξ_T^μ are restricted to the x-y plane, combined with some more changes of variables gives

$$
\begin{aligned}
M(q,q') &= -2s(2\pi)^4\,\delta(q+q')\int_{-\infty}^{\infty} d^2\boldsymbol{\xi}_\mathrm{T}\,\mathrm{e}^{-\mathrm{i}\mathbf{q}_\mathrm{T}\boldsymbol{\xi}_\mathrm{T}} \\
&\quad\times\left(\mathrm{e}^{\mathrm{i}\frac{\gamma(s)}{s}8\pi G\int\limits_{-\infty}^{\infty}d\sigma_1\int\limits_{-\infty}^{\infty}d\sigma_2\,D(\xi_\mathrm{T}^\mu+u_1^\mu\sigma_1-u_2^\mu\sigma_2)} - 1\right). \qquad (43)
\end{aligned}
$$

Here we introduce a graviton mass μ; then

$$
D(\xi_\mathrm{T}+u_1\sigma_1-u_2\sigma_2) = \int\frac{d^4k}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}k_\mu\xi_\mathrm{T}^\mu+\mathrm{i}k_\mu u_1^\mu\sigma_1-\mathrm{i}k_\mu u_2^\mu\sigma_2}\frac{1}{k^2+\mu^2-\mathrm{i}\epsilon}.
$$

Integration over $\int\limits_{-\infty}^{\infty} d\sigma_1 \int\limits_{-\infty}^{\infty} d\sigma_2$ yields two δ-functions:

$$\begin{aligned}\int\limits_{-\infty}^{\infty} d\sigma_1\, \mathrm{e}^{\mathrm{i}(ku_1)\sigma_1} &= 2\pi\,\delta(ku_1),\\ \int\limits_{-\infty}^{\infty} d\sigma_2\, \mathrm{e}^{\mathrm{i}(ku_2)\sigma_2} &= 2\pi\,\delta(ku_2).\end{aligned}$$

But $ku_1 = \frac{1}{\sqrt{2}}(k_3 - k^0)$, $ku_2 = \frac{1}{\sqrt{2}}(-k_3 - k^0)$, so that with the aid of the δ-functions we find:

$$k_3 = k^0 \quad \text{and} \quad k_3 = -k^0, \qquad \text{meaning} \quad k_3 = 0 = k^0.$$

These facts enable us to write for the exponential in Eq. (43):

$$\begin{aligned}&\mathrm{i}\frac{\gamma(s)}{s} 8\pi G \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k_\mu \xi^\mu_{\mathrm{T}}}}{k^2+\mu^2} (2\pi)^2\,\delta(ku_1)\,\delta(ku_2)\\ &= \mathrm{i}\frac{\gamma(s)}{s} 8\pi G \int \frac{d^2\mathbf{k}_{\mathrm{T}}}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}_{\mathrm{T}}\cdot\boldsymbol{\xi}^\mu_{\mathrm{T}}}}{\mathbf{k}^2_{\mathrm{T}}+\mu^2}.\end{aligned}$$

So far we have

$$M(q,q') = -2s(2\pi)^4\,\delta(q+q') \int_{-\infty}^{\infty} d^2\boldsymbol{\xi}_{\mathrm{T}}\, \mathrm{e}^{-\mathrm{i}\mathbf{q}_{\mathrm{T}}\boldsymbol{\xi}_{\mathrm{T}}} \left(\mathrm{e}^{\mathrm{i}\frac{\gamma(s)}{s} 8\pi G \int \frac{d^2k_{\mathrm{T}}}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}_{\mathrm{T}}\cdot\boldsymbol{\xi}^\mu_{\mathrm{T}}}}{\mathbf{k}^2_{\mathrm{T}}+\mu^2}} - 1 \right). \quad (44)$$

Noticing that

$$\frac{K_0(\mu|\mathbf{x}_{\mathrm{T}}|)}{2\pi} = \int \frac{d^2\mathbf{k}_{\mathrm{T}}}{(2\pi)^2} \frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}_{\mathrm{T}}\cdot\boldsymbol{\xi}^\mu_{\mathrm{T}}}}{\mathbf{k}^2_{\mathrm{T}}+\mu^2},$$

and introducing the T matrix via $M = \mathbb{1} - \mathrm{i}(2\pi)^4\,\delta(q+q')T$, we obtain

$$\mathrm{i}T = 2s \int d^2\mathbf{x}_{\mathrm{T}}\, \mathrm{e}^{-\mathrm{i}\mathbf{q}_{\mathrm{T}}\mathbf{x}_{\mathrm{T}}} \left(\mathrm{e}^{\mathrm{i}\frac{\gamma(s)}{s} 4G\, K_0(\mu|\mathbf{x}_{\mathrm{T}}|)} - 1 \right). \quad (45)$$

The integral over $\mathbf{x}_{\mathrm{T}}$ may be carried out, leading to

$$\mathrm{i}T = \frac{2\pi\sqrt{s(s-4m^2)}}{\mu^2} \frac{\Gamma\left(1 - \mathrm{i}2G\frac{\gamma(s)}{s}\right)}{\Gamma\left(\mathrm{i}2G\frac{\gamma(s)}{s}\right)} \left(\frac{4\mu^2}{k^2_{\mathrm{T}}}\right)^{1-\mathrm{i}2G\frac{\gamma(s)}{s}}, \quad (46)$$

or, introducing

$$\alpha(s) = G\frac{2\gamma(s)}{\sqrt{s(s-4m^2)}}, \tag{47}$$

$$\begin{aligned} \mathrm{i}T &= \frac{2\pi}{\mu^2}\sqrt{s(s-4m^2)}\,\frac{\Gamma\left(1-\mathrm{i}\alpha(s)\right)}{\Gamma\left(\mathrm{i}\alpha(s)\right)}\left(\frac{4\mu^2}{-t}\right)^{1-\mathrm{i}\alpha(s)}, \\ &= \mathrm{i}\frac{16\pi G\gamma(s)}{-t}\frac{\Gamma\left(1-\mathrm{i}\alpha(s)\right)}{\Gamma\left(1+\mathrm{i}\alpha(s)\right)}\left(\frac{4\mu^2}{-t}\right)^{-\mathrm{i}\alpha(s)}. \end{aligned} \tag{48}$$

This expression is evidently just the Born amplitude multiplied by

$$\frac{\Gamma\left(1-\mathrm{i}\alpha(s)\right)}{\Gamma\left(1+\mathrm{i}\alpha(s)\right)}\left(\frac{4\mu^2}{-t}\right)^{-\mathrm{i}\alpha(s)}.$$

Provided that $s>0$ (in fact we are interested in $s\to\infty$), and t is fixed with $\frac{t}{s}\to 0$, we may set $m=0$ in Eq. (47). This then brings us to 't Hooft's scattering amplitude [10] in terms of Lorentz invariant Mandelstam variables:

$$T(s,t) = \frac{8\pi G s^2}{-t}\frac{\Gamma(1-\mathrm{i}Gs)}{\Gamma(1+\mathrm{i}Gs)}\left(\frac{4\mu^2}{-t}\right)^{-\mathrm{i}Gs}. \tag{49}$$

The 't Hooft poles occur at $\mathrm{i}Gs = N$, $N=1,2,3,\dots$ These are the bound state poles in the $\frac{1}{r}$-potential of linearized gravity. This result had to be expected from similar calculations in QED. In fact all our results would have come out by just replacing eA^μ by $\frac{1}{2}p_\nu h^{\mu\nu}$ in the original action expression and Green's function equation for spin 1/2 QED. Hence the 't Hooft poles could have been already discovered by the end of the sixties, if one had not worried about the non-renormalizability of Einstein's theory and treated it instead as an effective field theory.

Note that nowhere in this article have we computed closed-loop processes, in particular, graviton loops. Hence our approximation is not able to produce any Callan–Symanzik β-function. Consequently we cannot generate any quantum correction to Newton's potential as has been done in more recent articles on the matter [11] [12]. But this does not come as a surprise since the eikonal approximation applied to QED is likewise unable to produce the Uehling potential or Lamb-shift. Finally we should point out that the 't Hooft poles originate from $|\mathbf{x}_\mathrm{T}|$, the impact parameter, reaching all the way down to zero, and thus from the short-distance behavior of the Newton interaction. If this is softened, as, for instance, in the string approach of graviton scattering, the poles disappear [13].

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